

Canonical volume forms on compact Kähler manifolds

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Abstract

We construct a canonical singular hermitian metric with semipositive curvature current on the canonical line bundle of a compact Kähler manifold with pseudoeffective canonical bundle. The method of the construction is a modification of the one in [T].

MSC: 14J15, 14J40, 32J18

1 Introduction

In [T], I have constructed the canonical singular hermitian metric on relative canonical bundles of projective families whose general fiber has pseudoeffective canonical bundles.

Although the construction in [T] works only for projective families, it is very likely that the similar result holds also for Kähler families. The purpose of this short article is to show that this is the case up to certain extent, i.e., the construction of the metric also works for the Kähler case, but we need more work to show the semipositivity of the curvature of the singular hermitian metrics on the families. The reason is that we cannot use the branched coverings in the case of Kähler families. To overcome this difficulty, it seems to be necessary to consider the variation of plurisubharmonic objects instead of holomorphic objects. This “horizontal semipositivity” of the curvature is very important in many applications such as the deformation invariance of plurigenera in the case of Kähler families.

Let X be a compact Kähler manifold of dimension n and let K_X be the canonical bundle of X . The purpose of this short note is to construct a canonical singular hermitian metric on K_X when K_X is pseudoeffective.

Theorem 1.1 *Let X be a compact Kähler manifold with pseudoeffective K_X . Then there exists a singular hermitian metric \tilde{h}_{can} on K_X such that*

1. \tilde{h}_{can} is uniquely determined by X .
2. \tilde{h}_{can} is an AZD of K_X , i.e., \tilde{h}_{can} has the following properties :
 - (a) The curvature current $\Theta_{\tilde{h}_{can}}$ of \tilde{h}_{can} is semipositive in the sense of current on X .

(b) For every $m \geq 0$,

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(\tilde{h}_{can}^m)) \simeq H^0(X, \mathcal{O}_X(mK_X))$$

holds for every $m \geq 0$.

□

Remark 1.2 If X is projective, \tilde{h}_{can} seems to be closely related to the super-canonical AZD \hat{h}_{can} in [T] (see Remark 2.4 below). □

We note that the bounded semipositive (n, n) form \tilde{h}_{can}^{-1} is an invariant volume form on X , i.e., the automorphism group of X preserves \tilde{h}_{can}^{-1} .

2 Construction of \tilde{h}_{can}

In this section, we shall prove Theorem 1.1.

2.1 The construction

The construction below is modeled after the one in [T].

Let X be a compact Kähler manifold. Let ω_0 be any Kähler form on X . For every $\varepsilon \in (0, 1]$, we set

$$h_\varepsilon := \inf\{h \mid h \text{ is a singular hermitian metric on } K_X, \Theta_h + \varepsilon \cdot \omega_0 \geq 0, \int_X h^{-1} = 1\},$$

where the infimum means the pointwise infimum. And set

$$dV_\varepsilon := h_\varepsilon^{-1}$$

And we set

$$\tilde{h}_{can} := \text{the lower envelope of } \liminf_{\varepsilon \downarrow 0} h_\varepsilon$$

and define

$$d\tilde{V}_{can} := \tilde{h}_{can}^{-1} = \text{the upper envelope of } \limsup_{\varepsilon \downarrow 0} dV_\varepsilon.$$

Then since the upper limit of a sequence of plurisubharmonic function locally uniformly bounded from above is again plurisubharmonic, if we take the upper envelope of the limit (cf. ([L, p.26, Theorem 5])), we see that \tilde{h}_{can} has semipositive curvature in the sense of current.

2.2 Upper estimate of dV_ε

Let $d\mu$ be a C^∞ volume form on X .

Lemma 2.1 *There exists a positive constant C independent of $\varepsilon \in (0, 1]$ such that*

$$dV_\varepsilon \leq C \cdot d\mu$$

holds on X . □

Proof of Lemma 2.1. Let x be a point on X . Let (U, z_1, \dots, z_n) be a local coordinate neighbourhood such such that

1. $z_1(x) = \dots = z_n(x) = 0$.
2. z_1, \dots, z_n are holomorphic on a neighbourhood of the closure of U .
3. U is biholomorphic to the unit open polydisk Δ^n in \mathbb{C}^n with center O via the coordinate (z_1, \dots, z_n) .

By the $\partial\bar{\partial}$ Poincaré lemma, there exists a positive C^∞ function φ_U such that

$$\omega_0|_U = \sqrt{-1}\partial\bar{\partial}\log\varphi_U$$

holds. By the construction, we may and do assume that φ_U can be taken so that φ is bounded on a neighbourhood of the closure of U .

Let dV be a bounded uppersemicontinuous semipositive (n, n) form such that

$$\sqrt{-1}\partial\bar{\partial}\log dV + \varepsilon\omega_0 \geq 0$$

and

$$\int_X dV = 1 \tag{1}$$

hold. We define the function a_U by

$$dV|_U = a_U \cdot (\sqrt{-1})^n \prod_{i=1}^n dz_i \wedge d\bar{z}_i$$

holds. Then since $\varepsilon \in (0, 1]$, we have that

$$\sqrt{-1}\partial\bar{\partial}\log(a_U \cdot e^{-\varphi_U}) \geq 0 \tag{2}$$

holds in the sense of current. And by (1)

$$\int_U e^{-\varphi_U} dV \leq \sup_U e^{-\varphi_U} \tag{3}$$

holds. Hence combining (2) and (3), by the submeanvalue property of plurisubharmonic functions, we see that

$$dV(x) \leq (\sup_U e^{-\varphi_U}) \cdot e^{\varphi_U(x)} \left(\frac{\sqrt{-1}}{2\pi}\right)^n \cdot \prod_{i=1}^n dz_i \wedge d\bar{z}_i$$

holds. Hence since X is compact, moving x , we obtain Lemma 2.1. \square

2.3 Lower estimate of dV_ε

The lower estimate is also easy. Let h be any singular hermitian metric on K_X with semipositive curvature current. We note that h^{-1} is a bounded semipositive (n, n) form on X . Then by the definition of $d\tilde{V}_\varepsilon$, we see that

$$dV_\varepsilon \geq \left(\int_X h^{-1}\right)^{-1} \cdot h^{-1}$$

holds. Hence letting ε tend to 0, we have the following lemma.

Lemma 2.2 *Let h be any singular hermitian metric on K_X with semipositive curvature current. Then we have that*

$$d\tilde{V}_{can} \geq (\int_X h^{-1})^{-1} \cdot h^{-1}$$

holds. \square

Lemma 2.2 immediately implies the following lemma.

Lemma 2.3 *\tilde{h}_{can} is an AZD of K_X , i.e.,*

1. $\Theta_{\tilde{h}_{can}}$ *is semipositive in the sense of current.*
2. *For every $m \geq 0$,*

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(\tilde{h}_{can}^m)) \simeq H^0(X, \mathcal{O}_X(mK_X))$$

holds. \square

Proof of Lemma 2.3. Let σ be a nonzero element of $H^0(X, \mathcal{O}_X(mK_X))$ for some $m \geq 0$. Then we see that

$$\frac{1}{|\sigma|^{\frac{2}{m}}} := \frac{h_0}{(h_0^m(\sigma, \sigma))^{\frac{1}{m}}}$$

is a singular hermitian metric on K_X with semipositive curvature current, where h_0 is any C^∞ hermitian metric on K_X . Hence by Lemma 2.2, we see that

$$\tilde{h}_{can} \leq |\int_X |\sigma|^{\frac{2}{m}}| \cdot \frac{1}{|\sigma|^{\frac{2}{m}}}.$$

This means that

$$\tilde{h}_{can}^m(\sigma, \sigma) \leq |\int_X |\sigma|^{\frac{2}{m}}|^m$$

holds. Hence

$$\sigma \in H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(\tilde{h}_{can}^m))$$

holds. Since σ is arbitrary, we complete the proof of Lemma 2.3. \square

Remark 2.4 *The above proof of Lemma 2.2 and [T, Lemma 2.5] implies that if X is projective, the supercanonical AZD \hat{h}_{can} of K_X constructed in [T] satisfies the inequalities*

$$\tilde{h}_{can} \leq \hat{h}_{can} \leq (\int_X \tilde{h}_{can}^{-1}) \cdot \tilde{h}_{can}$$

on X . \square

2.4 Independence from ω_0

Lemma 2.5 *$d\tilde{V}_{can}$ is independent of the choice of ω_0 .* \square

Proof of Lemma 2.5. Let ω'_0 be another Kähler form on X . Then since X is compact, there exists a positive constant $C_0 > 1$ such that

$$C_0^{-1} \cdot \omega'_0 \leq \omega_0 \leq C_0 \cdot \omega'_0$$

holds on X . For every $\varepsilon \in (0, 1]$, we set

$$h'_\varepsilon := \inf\{h \mid h \text{ is a singular hermitian metric on } K_X, \Theta_h + \varepsilon \cdot \omega'_0 \geq 0, \int_X h^{-1} = 1\},$$

where the infimum means the pointwise infimum. And set

$$dV'_\varepsilon := (h'_\varepsilon)^{-1}.$$

Then by the definitions of h_ε and dV_ε we have the inequalities :

$$h'_{C_0\varepsilon} \leq h_\varepsilon \leq h_{C_0^{-1}\varepsilon}$$

and

$$dV'_{C_0^{-1}\varepsilon} \leq dV_\varepsilon \leq dV_{C_0\varepsilon}$$

hold for every $\varepsilon \in (0, C_0^{-1}]$. Hence letting ε tend to 0, we see that $d\tilde{V}_{can}$ is independent of the choice of ω_0 . \square

2.5 Completion of the proof of Theorem 1.1

By Lemmas 2.1 and 2.2 and the construction of \tilde{h}_{can} , we see that \tilde{h}_{can} is a well defined singular hermitian metric on K_X . Also by Lemma 2.3, we see that \tilde{h}_{can} is an AZD of K_X . By Lemma 2.5, we see that \tilde{h}_{can} is independent of the choice of ω_0 .

Hence we complete the proof of Theorem 1.1. \square

2.6 A variant of the construction

One may modify the construction above. In fact one may define the singular hermitian metric without using the auxiliary Kähler form ω_0 .

$$\check{h}_{can} := \inf\{h \mid h \text{ is a singular hermitian metric on } K_X, \Theta_h \geq 0, \int_X h^{-1} = 1\}.$$

Then by the same proof as above, we may prove that \check{h}_{can} satisfies the same properties in Theorem 1.1. This construction is a little bit simpler. But it is not clear whether this construction has any advantage.

References

- [L] P. Lelong : Fonctions Plurisousharmoniques et Formes Differentielles Positives, Gordon and Breach (1968).
- [T] H. Tsuji : Canonical singular hermitian metrics on relative canonical bundles, mathArxiv. 0704.0566 (2007).

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